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**Abstract:** We define a new birth and death dynamics dealing with configurations of discs in the plane. We prove the convergence of the continuous process and propose a discrete scheme converging to the continuous case. This framework is developed to address image processing problems consisting in extracting objects. The derived algorithm is applied for tree crown extraction and bird detection from aerial images. The performance of this approach is shown on real data.

**Key-words:** Object extraction, Stochastic modeling, Birth and death dynamics

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# Extraction d'objets par une dynamique stochastique continue de naissance et mort

**Résumé :** Nous définissons une dynamique de naissance et mort s'appliquant à des configurations de disques dans le plan. Nous prouvons la convergence du processus continu et proposons une discrétisation du problème convergeant vers le cas continu. Cette approche est développée pour résoudre des problèmes d'analyse d'image liés à l'extraction d'objets. L'algorithme qui s'en déduit est appliqué aux problèmes de l'extraction de houppiers et à la détection d'oiseaux à partir d'images aériennes. Les performances de l'approche développée sont montrées sur des images réelles.

**Mots-clés :** Extraction d'objets, modélisation stochastique, dynamique de naissance/mort

# 1 Introduction

We propose a new stochastic algorithm to solve object extraction problems from images. The algorithm is based on an evolution of macro-objects in continuum. We consider here a model of possibly partially overlapping discs. Each disc in the final configuration is associated with a given object in the image, for example a tree or a bird. We define a stochastic evolution of a set of objects converging to the set of objects of interest in the image.

The evolution under consideration is a birth-and-death equilibrium dynamics on the configuration space of discs (or the configuration space of points) with a given stationary Gibbs measure (see [1, 2]). We define birth and death rates meeting the so-called detailed balance conditions. In our scheme the intensity of birth is a constant, whereas intensities of death depend on the energy function and the current configuration. This choice of rates has been made to optimize the convergence speed. Indeed, the volume of the space for birth is much bigger than the number of discs in the configuration. Each disc can be killed with a certain intensity depending on its neighborhood and the energy function. It is therefore faster to update the death map than the birth map.

We then embed the defined stationary dynamics into a simulated annealing procedure where the temperature of the system tends to zero in time. We thus obtain a non-stationary stochastic process, such that all weak limit measures have a support on configurations giving the global minimum of the energy function under a minimal number of discs in the configuration. The final step is the discretization of this non-stationary dynamics. The discretization is a non-homogeneous (in time and in space) Markov chain with transition probabilities depending on a temperature, the energy function and a discretization step. We prove that:

1) the discretization process converges to the continuous time process under fixed temperature as the step of discretization tends to zero;

2) if we apply the discretization process to any initial measure with a continuous density w.r.t. the Lebesgue-Poisson measure, then in the limit when the discretization step tends to 0, time tends to infinity and the temperature tends to 0, we get a measure concentrated on the global minima of the energy function with a minimal number of discs.

These results confirm that the proposed algorithm based on the discretization scheme together with the cooling procedure can be applied to problems of searching configurations giving global minima of the energy function.

We apply this framework to object detection from images. In some previous works, we have shown that marked point processes, defined by a Gibbs measure against the Poisson process, are adapted to such problems by modeling simple geometric objects defined by the marks associated to each point. Moreover, interactions between points allow to model some a priori information on the object configuration. This approach have been applied to detect different features such as road networks [3, 4], buildings [5] or trees [6]. In these references, the optimization of the defined marked point process is performed using a RJMCMC scheme [7]. In the RJMCMC scheme, each iteration consists in perturbating one or a couple of objects. Besides the rejection rate induces a huge computation time. In the proposed approach, each step concerns the whole configuration and there is no rejection. We thus obtained better performances in term of computational time, which allow to deal with real images of several millions of pixels. We build an energy function which embeds some a priori knowledge on the object configuration such as partial non overlapping between objects and a data term which allows the objects to fit the image under study. The optimization is then performed by using the proposed birth and death dynamics. Some results are shown on real data for the problems of tree and bird detection.

# 2 Description of the model

## 2.1 Configuration space

We consider finite systems of discs $\{d_{x_1}, \ldots, d_{x_k}\}$ of the same radius $r$ with a hard core distance $\epsilon$ between any two elements, lying in a bounded domain $V \subset R^2$. Let

$$\gamma = \{x_i\} \in \Gamma_d(V), \quad x_i \in V \subset R^2,$$

be a configuration of the centers of discs and where $\Gamma_d(V)$ denotes the configuration space of the discs center in $V$. Since the domain $V$ is bounded the number of discs in any configuration is uniformly bounded

$$|\gamma| \ < \ N \ = \ \frac{4|V|}{\pi\epsilon^2},$$

where $|V|$ is the volume of $V$. The set $\Gamma_d(V)$ can be decomposed into strata:

$$\Gamma_d(V) \ = \ \bigcup_{n=0}^{N} \Gamma_d(V,n),$$

where each stratum $\Gamma_d(V,n)$ is the set of configurations containing $n$ discs, and $\Gamma_d(V,0) = \{\emptyset\}$. Since we consider unordered sets of discs, the set $\Gamma_d(V,n)$ for any $n > 0$ can be represented as a factor set

$$\Gamma_d(V,n) \ = \ V_d^n \ / \ S_n,$$

where

$$V_d^n = \{(x_1, \ldots, x_n) \in V^n : \ |x_i - x_j| \geq \epsilon, \ i,j = 1, \ldots, n, \ i \neq j\},$$

and $S_n$ is the permutation group in the set $(x_1, \ldots, x_n)$. We define a mapping

$$\Pi_n : V_d^n \ \to \ \Gamma_d(V,n), \quad \Pi_n(x_1, \ldots, x_n) \in \Gamma_d(V,n) \tag{1}$$

Each function $F(\gamma), \ \gamma \in \Gamma_d(V)$ on the space $\Gamma_d(V)$ can be represented as a function in the Fock space:

$$F_0, \ F_1(x_1), \ F_2(x_1,x_2), \ldots, F_N(x_1, \ldots, x_N), \tag{2}$$

where $F_0 = F(\emptyset), \ F_n(x_1, \ldots, x_n) = F(\Pi_n(x_1, \ldots, x_n))$ is a symmetrical function on $V_d^n$. A function $F$ on the space $\Gamma_d(V)$ is said to be a smooth (continuous) function if each function in (2) is a smooth (continuous) function on $V_d^n$.

Let us consider a measure in the space $\Gamma_d(V,n)$:

$$\lambda_n(A) \;=\; \frac{\Pi_n^{-1}(A)}{n!}, \quad A\subset\Gamma_d(V,n),\ \lambda_0(\emptyset)=1. \tag{3}$$

Here $|\Pi_n^{-1}(A)|$ is the $2n$-dimensional Lebesgue volume of the domain $\Pi_n^{-1}(A)\subset V_d^n$ (a complete preimage of $A$ under mapping $\Pi_n$). The measure $\lambda$ on the space $\Gamma_d(V)$, such that the restriction of $\lambda$ on each stratum $\Gamma_d(V,n)$ is given by $\lambda_n$, is called the Lebesgue-Poisson measure.

## 2.2 Energy function

We define on the space $\Gamma_d(V)$ a real-valued smooth and bounded from below function $H(\gamma)$ which is called the energy function. We set $H(\emptyset)=0$. Then the corresponding Fock representation for $H$ has the form

$$H \;=\; (0,\ H_1(x_1),\ H_2(x_1,x_2),\dots,H_N(x_1,\dots,x_N))\,. \tag{4}$$

The Gibbs distribution $\mu_\beta^V$ on the space $\Gamma_d(V)$ generated by the energy $H(\gamma)$ is defined by the density $p_V(\gamma)=\frac{d\mu_\beta^V}{d\lambda}(\gamma)$ with respect to the Lebesgue-Poisson measure $\lambda$:

$$p_V(\gamma) \;=\; \frac{z^{|\gamma|}}{Z_{\beta,V}}\ \exp\{-\beta H(\gamma)\}, \tag{5}$$

with positive parameters $\beta>0,\ z>0$ and a normalizing factor $Z_{\beta,V}$:

$$Z_{\beta,V}=\int\limits_{\Gamma_d(V)} z^{|\gamma|}\exp\{-\beta H(\gamma)\}d\lambda(\gamma)=1+\sum_{n=1}^{N}\frac{z^n}{n!}\int\limits_{V_d^n} e^{-\beta H_n(x_1,\dots,x_n)}dx_1\dots dx_n.$$

We formulate now some assumptions on the energy function $H_V(\gamma)$. Denote by

$$\bar H \;=\; \min_{\gamma\in\overline{\Gamma_d(V)}} H(\gamma),$$

where $\overline{\Gamma_d(V)}$ is the closure of $\Gamma_d(V)$, and let

$$T_V \;=\; \{\bar\gamma\in\overline{\Gamma_d(V)}:\ H(\bar\gamma)=\bar H\}$$

be a set of all points from $\overline{\Gamma_d(V)}$ giving the global minimum $\bar H$ of the function $H(\gamma)$. The set $T_V$ can be written as

$$T_V \;=\; \bigcup_{n=0}^{N} T_{V,n},$$

where $T_{V,n}$ is a set of configurations from $T_V$ which are also configurations from $\Gamma_d(V,n)$, i.e. contain exactly $n$ discs.

In practice, this energy contains a first term representing a priori knowledge on the discs configuration and which is defined by interactions between neighboring discs, and a second term, obtained from data, which is defined for each object and which can be negative.

# 3 Convergence of measures

We assume that
1) the set $T_V$ is finite and situated in $\Gamma_d(V)$,
2) for any configuration $\bar\gamma \in T_{V,n}$ any preimage $(\bar x_1,\dots,\bar x_n) \in \Pi_n^{-1}(\bar\gamma)$ of $\bar\gamma$ is a non-degenerated critical point of the function $H_n$, i.e.:

$$\frac{\partial H_n}{\partial y_i^m}(\bar x_1,\dots,\bar x_n) \;=\; 0,$$

for any $i = 1,\dots,n,\ \bar x_i = (y_i^1, y_i^2) \in V,\ m = 1,2$, and the matrix

$$\mathcal{A}\;(\bar\gamma) \;=\; \left\{ \frac{\partial^2 H_n}{\partial y_i^{m_1}\partial y_j^{m_2}}(\bar x_1,\dots,\bar x_n) \right\}$$

at point $(\bar x_1,\dots,\bar x_n)$ is strictly positive-definite (this matrix is the same for all preimages of the configuration $\bar\gamma$). We denote

$$B(\bar\gamma) \;=\; det\ \mathcal{A}(\bar\gamma). \tag{6}$$

**Theorem 1.** *Let $n_0 \in [0,\dots,N]$ be the minimal index for which the set $T_{V,n}$ is not empty. Then the Gibbs distributions $\mu_\beta$ converge weakly as $\beta \to \infty$ to a distribution $\mu_\infty$ on $\Gamma_d(V)$ of the form*

$$\mu_\infty \;=\; \sum_{\bar\gamma \in T_{V,n_0}} C_{\bar\gamma}\delta_{\bar\gamma}\ \textit{if}\ n_0 > 0, \quad \textit{and} \quad \mu_\infty \;=\; \delta_{\{\emptyset\}}\ \textit{if}\ n_0 = 0. \tag{7}$$

*Here $\delta_{\bar\gamma}$ is the unit measure concentrated on the configuration $\bar\gamma$, and the coefficients $C_{\bar\gamma}$ hold the equality*

$$\sum_{\bar\gamma \in T_{V,n_0}} C_{\bar\gamma} \;=\; 1.$$

**Proof.** Let $F(\gamma)$ be a smooth function on $\Gamma_d(V)$. Then

$$\langle F\rangle_{\mu_\beta} \;=\; \frac{I(F)}{Z_{\beta,V}} \;= \tag{8}$$

$$Z_{\beta,V}^{-1}\left(F_0 \;+\; \sum_{n=1}^{N}\frac{z^n}{n!}\int\limits_{V_d^n} F_n(x_1,\ldots,x_n)\; e^{-\beta H_n(x_1,\ldots,x_n)}\; dx_1\ldots dx_2\right)\;.$$

Let us consider each integral in (8)

$$I_n(F_n) \;=\; \int\limits_{V_d^n} F_n(x_1,\ldots,x_n)\; e^{-\beta H_n(x_1,\ldots,x_n)}\; dx_1\ldots dx_2$$

separately. If the set $T_{V,n}$ is empty, then the integral $I_n(F_n)$ meets the estimate

$$|I_n(F_n)| \;<\; e^{-\beta h_n}|V_d^n|, \tag{9}$$

where

$$h_n \;=\; \min H_{V,n}(x_1,\ldots,x_n) \;>\; \bar{H}.$$

If $T_{V,n}$ is not empty, then the following asymptotics holds, see for example [8]

$$I_n(F_n) \;=\; e^{-\beta\bar{H}}\left(\sum_{\bar{\gamma}\in T_{V,n}} F(\bar{\gamma})R(\bar{\gamma})\beta^{-n/2} \;+\; \beta^{-n/2-1/2}S(F,\beta)\right), \tag{10}$$

where $S(F,\beta)$ is bounded as $\beta\to\infty$, and

$$R(\bar{\gamma}) \;=\; \frac{1}{(2\pi)^{n/2}}B^{-1/2}(\bar{\gamma}),$$

where $B(\bar{\gamma})$ is defined in (6). Then bound (9) and asymptotics (10) imply that for $n_0>0$ and $\beta\to\infty$ we get

$$I(F) \;=\; F_0 \;+\; \sum_{n=1}^{N}\frac{z^n}{n!}I_n(F_n) \;=\; \frac{e^{-\beta\bar{H}}}{\beta^{n_0/2}}\left(\sum_{\bar{\gamma}\in T_{V,n_0}} F(\bar{\gamma})R(\bar{\gamma}) \;+\; o(1)\right). \tag{11}$$

Analogously,

$$Z_{\beta,V} \;=\; I(1) \;=\; \frac{e^{-\beta\bar{H}}}{\beta^{n_0/2}}\left(\sum_{\bar{\gamma}\in T_{V,n_0}} R(\bar{\gamma}) \;+\; o(1)\right). \tag{12}$$

Finally from (8), (11) and (12) we get (7) with

$$C(\bar{\gamma}) \;=\; \frac{R(\bar{\gamma})}{\sum_{\bar{\gamma}\in T_{V,n_0}} R(\bar{\gamma})}.$$

The case $n_0=0$ can be studied in the same way. Since the space of smooth functions is dense in the space of bounded functions $B(\Gamma_d(V))$, we prove the weak convergence of measures $\mu_\beta\to\mu_\infty$ on the space $B(\Gamma_d(V))$. □

# 4 A continuous-time equilibrium dynamics

We consider an operator in the space $C(\Gamma_d(V))$ of continuous bounded functions on $\Gamma_d(V)$ of the form:

$$(L_\beta\ f)(\gamma) \ = \ \sum_{x\in\gamma} e^{\beta E(x,\gamma\backslash x)}(f(\gamma\backslash x) - f(\gamma)) \ + \ z \int\limits_{V(\gamma)} (f(\gamma \cup y) - f(\gamma))\ dy, \tag{13}$$

where

$$V(\gamma) \ = \ V\backslash D(\gamma), \quad D(\gamma) \ = \ (\cup_{x\in\gamma}\mathcal{B}_x(\epsilon)) \cap V,$$

where $\mathcal{B}_x(\epsilon)$ is the disk with center at point $x \in V$ and radius $\epsilon$, and

$$E(x, \gamma\backslash x) \ = \ H(\gamma) - H(\gamma\backslash x).$$

The operator defined in equation (13) is the generator of a birth-and-death process in the domain $V \subset R^2$ with birth intensity $b(\gamma, x)$ in the unordered configuration $\gamma$ at $x$ and death intensity $d(\gamma\backslash x, x)$ from the configuration $\gamma$ at position $x$ respectively given by :

$$b(\gamma, x)\ dx \ = \ z\ dx, \quad d(\gamma\backslash x, x) \ = \ e^{\beta E(x,\gamma\backslash x)}.$$

Under this choice of the birth and death intensities, the detailed balance condition holds:

$$\frac{b(\gamma, x)}{d(\gamma\backslash x, x)} \ = \ \frac{p_V(\gamma)}{p_V(\gamma\backslash x)} \ = \ z\ e^{-\beta E(x,\gamma\backslash x)},$$

and consequently, see for example [9], the corresponding birth-and-death process associated with the stochastic semigroup $T_\beta(t) = e^{tL_\beta}$ is time reversible, and its equilibrium distribution is the Gibbs stationary measure $\mu_\beta^V$ with density (5).

**Theorem 2.** *1) The operator $L_\beta$ is a bounded operator in the space of bounded functions $B(\Gamma_d(V))$ and in $L_2(\Gamma_d(V), \mu_\beta)$, moreover $L_\beta$ is a self-adjoint operator in $L_2(\Gamma_d(V), \mu_\beta)$.*
*2) The family of operators $T_\beta(t) = e^{tL_\beta}$, $t \geq 0$ forms a self-adjoint Markov semigroup in $L_2(\Gamma_d(V), \mu_\beta)$, i.e.*

$$e^{tL_\beta}1 = 1, \quad \textit{and} \quad e^{tL_\beta}F \geq 0 \ \ \textit{for any non-negative} \ \ F \in L_2(\Gamma_d(V), \mu_\beta). \tag{14}$$

*3) The semigroup $T_\beta(t)$ is a contraction semigroup in $B(\Gamma_d(V))$.*
*4) The semigroup $T_\beta(t)$ meets the condition of improving positivity, i.e. for any non-negative function $F \geq 0$ we get $T_\beta(t)F > 0$*

For the proof, see Appendix A.

**Corollary.** The improving positivity property implies that there exists a unique fixed vector of the operators $T_\beta(t) = e^{tL_\beta}$ in $L_2(\Gamma_d(V), \mu_\beta)$ (which is equal to 1), see [10].

The convergence to the stationary measure $\mu_\beta$ is guaranteed by the general result given by C. Preston in [11]. We consider a family $\mathbb{B}(\lambda)$ of measures $\nu$ on the space $\Gamma_d(V)$ with a bounded density $\tilde{p}_\nu(\gamma)$ with respect to Lebesgue-Poisson measure $\lambda$. This, in particular, implies that a density $p_\nu(\gamma)$ of the measure $\nu \in \mathbb{B}(\lambda)$ w.r.t. a Gibbs measure $\mu_\beta$ (for any $\beta$):

$$p_\nu(\gamma) \;=\; \frac{d\nu}{d\mu_\beta}(\gamma)$$

is also bounded, and consequently, $p_\nu(\gamma) \in L_2(\Gamma_d(V), \mu_\beta)$. Then we can define the evolution $\nu_t \equiv T(t)\nu$ of the measure $\nu \in \mathbb{B}(\lambda)$ as follows:

$$\langle \nu_t, F\rangle \;=\; (T_\beta(t)p_\nu, F)_{\mu_\beta}.$$

Notice that property 2) of Theorem 2 implies that $T_\beta(t)p_\nu$ is again a density w.r.t. a Gibbs measure, i.e.

$$T_\beta(t)p_\nu \geq 0, \quad \langle T_\beta(t)p_\nu\rangle_{\mu_\beta} = \langle p_\nu\rangle_{\mu_\beta} = 1.$$

**Theorem 3.** *Let $\nu \in \mathbb{B}(\lambda)$. Then for any $F \in L_2(\Gamma_d(V), \mu_\beta)$ we get*

$$\langle T_\beta(t)\nu, F\rangle \;\equiv\; \langle \nu_t, F\rangle \;=\; (T_\beta(t)p_\nu, F)_{\mu_\beta} \;\rightarrow\; \langle F\rangle_{\mu_\beta}. \tag{15}$$

The proof of theorem 3 follows from the general theorems by C. Preston [11].

# 5 Approximation process

In this section, we define a discrete time approximation of the proposed continuous birth and death process.

We consider Markov chains $T_{\beta,\delta}(n), n = 0, 1, 2, \ldots$ on the same space $\Gamma_d(V)$. The process $T_{\beta,\delta}(n)$ can be described as follows: a configuration $\gamma$ is transformed to a configuration $\gamma' = \gamma_1 \cup \gamma_2$, where $\gamma_1 \subseteq \gamma$, and $\gamma_2$ is a configuration of centers of discs such that $\gamma_1 \cap \gamma_2 = \emptyset$ and is distributed w.r.t. the Poisson law with intensity $z$.

This transformation embed a birth part given by $\gamma_2$ and a death part given by $\gamma\backslash\gamma_1$.

The transition probability for the death of a particle at $x$ (i.e. a disc with the center at $x$) from the configuration $\gamma$ is given by:

$$p_{x,\delta} \;=\; \begin{cases} \frac{e^{\beta E(x,\gamma\backslash x)}\;\delta}{1\;+\;e^{\beta E(x,\gamma\backslash x)}\;\delta} \;=\; \frac{a_x\delta}{1+a_x\delta}, & \text{if} \quad \gamma \rightarrow \gamma\backslash x, \\ \\ \frac{1}{1+a_x\delta}, \quad \text{if} \quad \gamma \rightarrow \gamma \quad (x \text{ survives}). \end{cases} \tag{16}$$

with $a_x = a_x(\gamma) = e^{\beta E(x,\gamma\backslash x)}$. Moreover, all the particles are killed independently, and both configurations $\gamma_1$ and $\gamma_2$ are independent.

The transitions associated with the birth of a new particle in a small domain $\Delta y \subset V(\gamma)$ have the following probability distribution:

$$q_{y,\delta} \;=\; \begin{cases} z\Delta y\delta, \quad \text{if} \quad \gamma \rightarrow \gamma \cup y, \\ \\ 1 - z\Delta y\delta, \quad \text{if} \quad \gamma \rightarrow \gamma \quad (\text{ no birth in } \Delta y). \end{cases} \tag{17}$$

Finally, the transition operator $P_{\beta,\delta}$ for the process

$$T_{\beta,\delta}(n) \; = \; P^n_{\beta,\delta}$$

has the following form:

$$(P_{\beta,\delta} f)(\gamma) \; = \; \sum_{\gamma_1 \subseteq \gamma} \prod_{x \in \gamma_1} \frac{1}{1 + a_x \delta} \prod_{x \in \gamma \setminus \gamma_1} \frac{a_x \delta}{1 + a_x \delta} \tag{18}$$
$$\Xi_\delta^{-1}(\gamma_1) \sum_{k=0}^{\infty} \int\limits_{V_k(\gamma_1)} \frac{(z\delta)^k}{k!} f(\gamma_1 \cup y_1 \cup \ldots \cup y_k) \; dy_1 \ldots dy_k,$$

where $\Xi_\delta(\gamma) \; = \; \Xi_\delta(V(\gamma), z, \delta)$ is the normalizing factor for the conditional Lebesgue-Poisson measure under a given configuration of discs $\gamma_1$. We prove below that the approximation process $T_{\beta,\delta}(t) \equiv T_{\beta,\delta}\left(\left[\frac{t}{\delta}\right]\right)$ converges to the continuous time process $T_\beta(t)$ uniformly on bounded intervals $[0, \bar{t}]$ as the discretization step $\delta$ tends to 0.

Let us denote $\mathcal{L} = B(\Gamma_d(V))$ a Banach space of bounded functions on $\Gamma_V$ with a norm

$$\|F\| \; = \; \sup_{\gamma \in \Gamma_d(V)} |F(\gamma)|.$$

**Theorem 4.** *For each* $F \in \mathcal{L}$

$$\|T_{\beta,\delta}(t)F \; - \; T_\beta(t)F\|_{\mathcal{L}} \; = \; \sup_\gamma |(T_{\beta,\delta}(t)F)(\gamma) \; - \; (T_\beta(t)F)(\gamma)| \; \to \; 0, \tag{19}$$

*as* $\delta \to 0$ *for all* $t \geq 0$ *uniformly on bounded intervals of time.*

See Appendix B for the proof.

**Corollary.** The result of theorem 4 implies that for any $F, G \in B(\Gamma_d(V))$ we get

$$(G, T_{\beta,\delta}(t)F)_{\mu_\beta} \; \to \; (G, T_\beta(t)F)_{\mu_\beta} \quad \text{as} \quad \delta \to 0. \tag{20}$$

We denote by $S_{\beta,\delta}(n)$ an adjoint to $T_{\beta,\delta}(n)$ semigroup acting on measures, such that for any $\nu \in \mathbb{B}(\lambda)$:

$$\langle S_{\beta,\delta}(n)\nu, F\rangle = (p_\nu, T_{\beta,\delta}(n)F)_{\mu_\beta}.$$

We now formulate the main result about convergence.

**Main theorem.** *Let* $F \in B(\Gamma_d(V))$ *and an initial measure* $\nu \in \mathbb{B}(\lambda)$. *Then under relation*

$$\delta \; e^{\beta b} \; < \; const \tag{21}$$

*with* $b = \sup_{\gamma \in \Gamma_d(V)} \; \sup_{x \in \gamma} E(x, \gamma \setminus x)$ *we have*

$$\lim_{\beta \to \infty, \; t \to \infty, \; \delta \to 0} \langle F \rangle_{S_{\beta,\delta}([\frac{t}{\delta}])\nu} \; = \; \langle F \rangle_{\mu_\infty}, \tag{22}$$

where measure $\mu_\infty$ is defined in Theorem 1, and $\langle F\rangle_{S_{\beta,\delta}([\frac{t}{\delta}])\nu} = \langle S_{\beta,\delta}([\frac{t}{\delta}])\nu, F\rangle$.

**Proof.** We can write as follows

$$\langle F\rangle_{S_{\beta,\delta}([\frac{t}{\delta}])\nu} - \langle F\rangle_{\mu_\infty} = (p_\nu, T_{\beta,\delta}([\frac{t}{\delta}])F)_{\mu_\beta} - (p_\nu, T_\beta(t)F)_{\mu_\beta} +$$

$$+(T_\beta(t)p_\nu, F)_{\mu_\beta} - \langle F\rangle_{\mu_\beta} + \langle F\rangle_{\mu_\beta} - \langle F\rangle_{\mu_\infty}.$$

Then, using the results of theorem 1 and the limit relations (7), (15) and (20) we get (22). In addition, relation (21) follows from the approximation technique (see appendix B, equations (43) and (50)).

**Remark.** Relation (22) determines the limit over three quantities: $\beta \to \infty,\ t \to \infty, \delta \to 0$. In the approximation technique we used the relation (21) between $\delta$ and $\beta$:

$$\delta = \phi(\beta)\ e^{-\beta b} \quad \text{with} \quad \phi(\beta) = O(1) \ \text{ as } \ \beta \to \infty.$$

Unfortunately we could not find the relation between $t$ and $\beta$. If we have the relation $t = \psi(\beta)$ in an explicit form, then (22) can be rewritten as a limit when $t \to \infty$ under two relations $\beta(t) = \psi^{-1}(t)$ and $\delta(t) = \phi(\beta(t))e^{-\beta(t)b}$.

# 6 Application to object detection from numerical images

## 6.1 Model

Let consider a numerical image on the lattice $I \subset Z^2$, defined as follows:

$$\begin{aligned} Y : I &\rightarrow \Lambda \subset N \\ s &\mapsto y_s \end{aligned} \tag{23}$$

Each $y_s$ refers to the grey level at pixel $s$, on the lattice $I = \{1, \cdots, NL\} \times \{1, \cdots, NC\}$, $NL$ (resp. $NC$) being the number of lines (resp. columns) of the analyzed image. We consider configurations of centers of discs $\gamma = \{x_i\} \in \Gamma_d(V)$, where $V = [1/2, NL+1/2] \times [1/2, NC+1/2]$. Each disk in the final configuration represents an object in the image. The hard core distance $\epsilon$ is naturally taken to be equal to the data resolution: $\epsilon = 1$ pixel. To define the energy $H$, we first consider some prior knowledge. We want to minimize the overlap between objects. However, to obtain a more flexible model w.r.t. the data and the kind of objects, we do not forbid but only penalize overlapping objects. We define a pairwise interaction as follows:

$$\forall \{x_i, x_j\} \in \gamma \times \gamma, H_2(x_i, x_j) = \max\left(0, 1 - \frac{||x_j - x_i||}{2r}\right) \tag{24}$$

where $||.||$ is the Euclidean norm and $r$ is the radius of the underlying disc.

A first order term is then added for each object to fit the disc configuration onto the data. We consider that there is an object, modeled by a disc centered at pixel $s$, in the image, if the grey level values of the pixels inside the projection of the disc onto the lattice are statistically different from those of the pixels in the neighborhood of the disc. To quantify this difference we compute the Bhattacharya distance between the associated distributions. Denote $D_1(s)$, the projection of the disc with radius $r$ centered at $s$ onto the lattice, and $D_2(s)$ the surrounding crown:

$$D_1(s) = \{t \in I : ||t-s|| \leq r\} \text{ and } D_2(s) = \{t \in I : ||t-s|| \leq r+1\} \backslash D_1(s). \tag{25}$$

We consider the mean and the variance of the data of these two subsets:

$$\begin{aligned} \mu_1(s) &= \frac{\sum_{t\in D_1(s)} y_t}{\sum_{t\in D_1(s)} 1} \quad \text{and} \quad \mu_2(s) = \frac{\sum_{t\in D_2(s)} y_t}{\sum_{t\in D_2(s)} 1} \\ \sigma_1^2(s) &= \frac{\sum_{t\in D_1(s)} y_t^2}{\sum_{t\in D_1(s)} 1} - \mu_1(s)^2 \quad \text{and} \quad \sigma_2^2(s) = \frac{\sum_{t\in D_2(s)} y_t^2}{\sum_{t\in D_2(s)} 1} - \mu_2(s)^2 \end{aligned} \tag{26}$$

Assuming Gaussian distributions, the Bhattacharya distance between the distributions in $D_1(s)$ and in $D_2(s)$ is then given by:

$$B(s) = \frac{1}{4}\left(\mu_1(s) - \mu_2(s)\right)^2 \sqrt{\sigma_1^2(s) + \sigma_2^2(s)} - \frac{1}{2}\log\frac{2\sigma_1(s)\sigma_2(s)}{\sigma_1^2(s) + \sigma_2^2(s)}. \tag{27}$$

From this distance between the two distributions, a first order energy term is built:

$$\forall x_i \in \gamma,\ H_1(x_i) = \begin{cases} \left(1 - \frac{B(i)}{T}\right) & \text{if} \quad B(i) < T \\ \left(\exp -\frac{B(i)-T}{3B(i)} - 1\right) & \text{if} \quad B(i) \geq T \end{cases} \tag{28}$$

where $i$ is the closest point to $x_i$ on the lattice, and $T$ is a threshold parameter. Finally, under the hard core constraint, the global energy is written as follows:

$$H(\gamma) = \alpha \sum_{x_i\in\gamma} H_1(x_i) + \sum_{\{x_i,x_j\}\in\gamma\times\gamma, i\neq j} H_2(x_i, x_j) \tag{29}$$

where $\alpha$ is a weighting parameter between the data term and the prior.

## 6.2 Algorithm

The algorithm simulating the process is defined as follows:

- **Computation of the first order term:** For each site $s \in I$ compute $H_1(s)$ from the data
- **Computation of the birth map:** To speed up the process, we consider a non homogeneous birth rate to favor birth where the first order term is low (i.e. where the data tend to define an object):

$$\forall s \in I, b(s) = 1 + 9\frac{\max_{t\in I} H_1(t) - H_1(s)}{\max_{t\in I} H_1(t) - \min_{t\in I} H_1(t)}. \tag{30}$$

The normalized birth rate is then given by:

$$\forall s \in I, B(s) = \frac{zb(s)}{\sum_{t \in I} b(s)} \tag{31}$$

This non homogeneous birth rate refers to a non homogeneous reference Poisson measure. It has no impact on the convergence to the global minima of the energy function but do have an impact on the speed of convergence in practice by favoring birth in relevant locations.

- **Main program:** initialize the inverse temperature parameter $\beta = \beta_0$ and the discretization step $\delta = \delta_0$ and alternate birth and death steps
  - **Birth step:** for each $s \in S$, if $x_s = 0$ (no point in $s$) add a point in $s$ ($x_s = 1$) with probability $\delta B(s)$ (note that the hard core constraint with $\epsilon = 1$ pixel is satisfied).
  - **Death step:** consider the configuration of points $x = \{s \in I : x_s = 1\}$ and sort it from the highest to the lowest value of $H_1(s)$. For each point taken in this order, compute the death rate as follows:

    $$d_x(s) = \frac{\delta a_x(s)}{1 + \delta a_x(s)}, \tag{32}$$

    where:

    $$a_x(s) = \exp -\beta \left( H(x/\{x_s\}) - H(x) \right). \tag{33}$$

    and kill $s$ ($x_s = 0$) with probability $d(s)$.
  - **Convergence test:** if the process has not converged, decrease the temperature and the discretization step by a given factor and go back to the birth step. The convergence is obtained when all the objects added during the birth step, and only these ones, have been killed during the death step.

## 6.3 Results

The first application we address concerns tree crown extraction from aerial images. We consider 50cm resolution images of poplars. Some examples of the obtained results are given on figures 1 and 2. The results are satisfactory. One can remark a few false alarms on figure 1, on the border on the plantation, due to shadows and a few misdetection on figure 2 on small trees for which the chosen radius (3 pixels) is too big.

The second application concerns the counting of flamingo population. An extract of the obtained result is given on figure 3 for the initial image and on figure 4 for the detected birds. Almost all the birds have been correctly detected. The full image contains $6128 \times 3920$ pixels and has been analyzed in ten minutes on a bi-processor 2GHz PC. This represents a main advantage with respect to more standard optimization techniques based on a RJMCMC sampler [6, 12]. Indeed, the speed of convergence and the computational efficiency of the proposed algorithm allow us to deal with with huge sets of data in a reasonable time.

# 7 Conclusion

In this report, we have proposed a new approach for detecting objects in an image. This approach is based on a birth and death process. We have proven the convergence of the continuous process. We then have described a discretization scheme and proven its convergence to the continuous process. From this general framework, we have proposed a disc model which permits the detection of objects in a given image. Two applications, concerning tree and bird detection, have shown the relevence of the proposed approach. The two main advantages of this technique are its generality and its computational efficiency.

Next steps will concern the generalization of the model to a broader class of objects. Taking into account other kinds of objects such as ellipses or rectangles is straightforward. However, it will be interesting to embed some randomness in the definition of objects. Dealing with random radius or more generally random marks associated with the points in the configuration will increase the application domain of this promising approach. To tackle this new generation of models, we are currently working of new dynamics for addressing geometric changes in the configuration such as object dilation, translation, rotation or object splitting and merging.

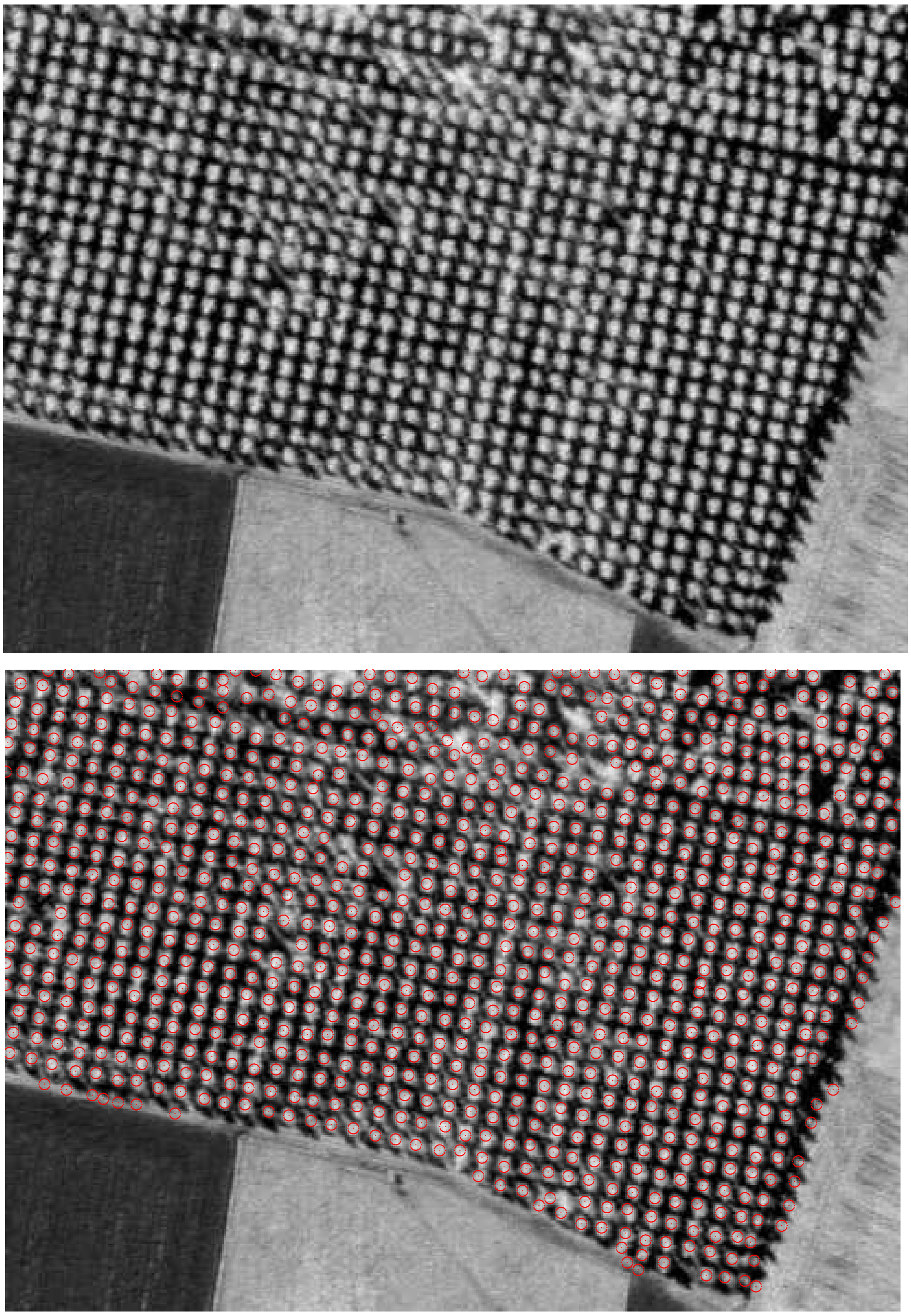

Figure 1: Result on a poplar plantation (top: initial image © IFN, bottom: detected trees)

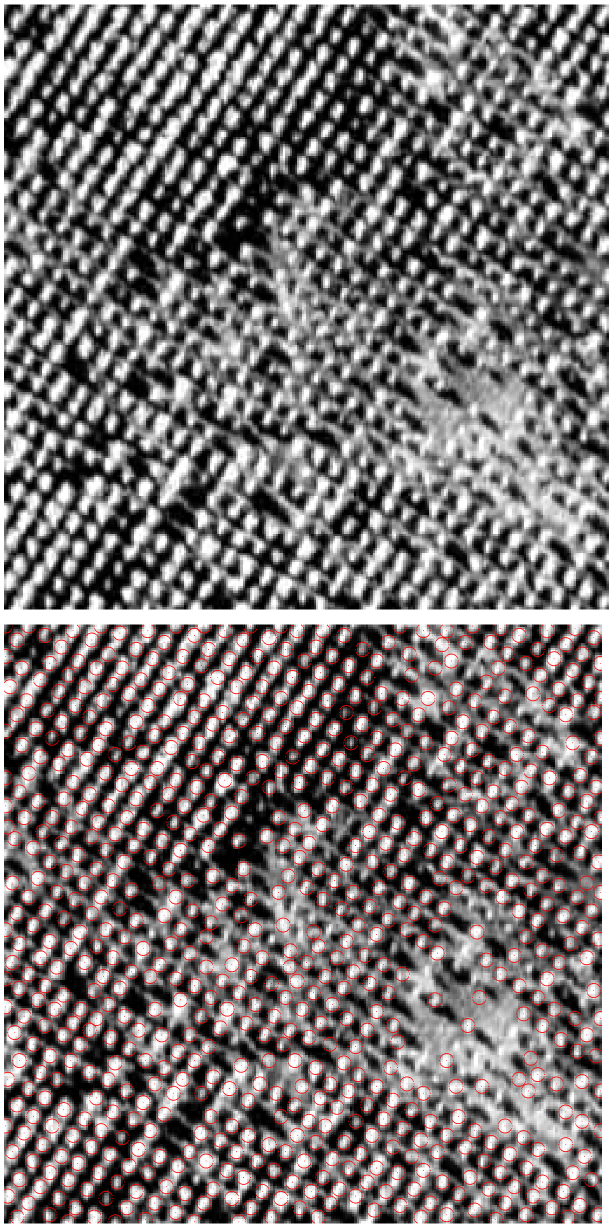

Figure 2: Result on a poplar plantation (top: initial image © IFN, bottom: detected trees)

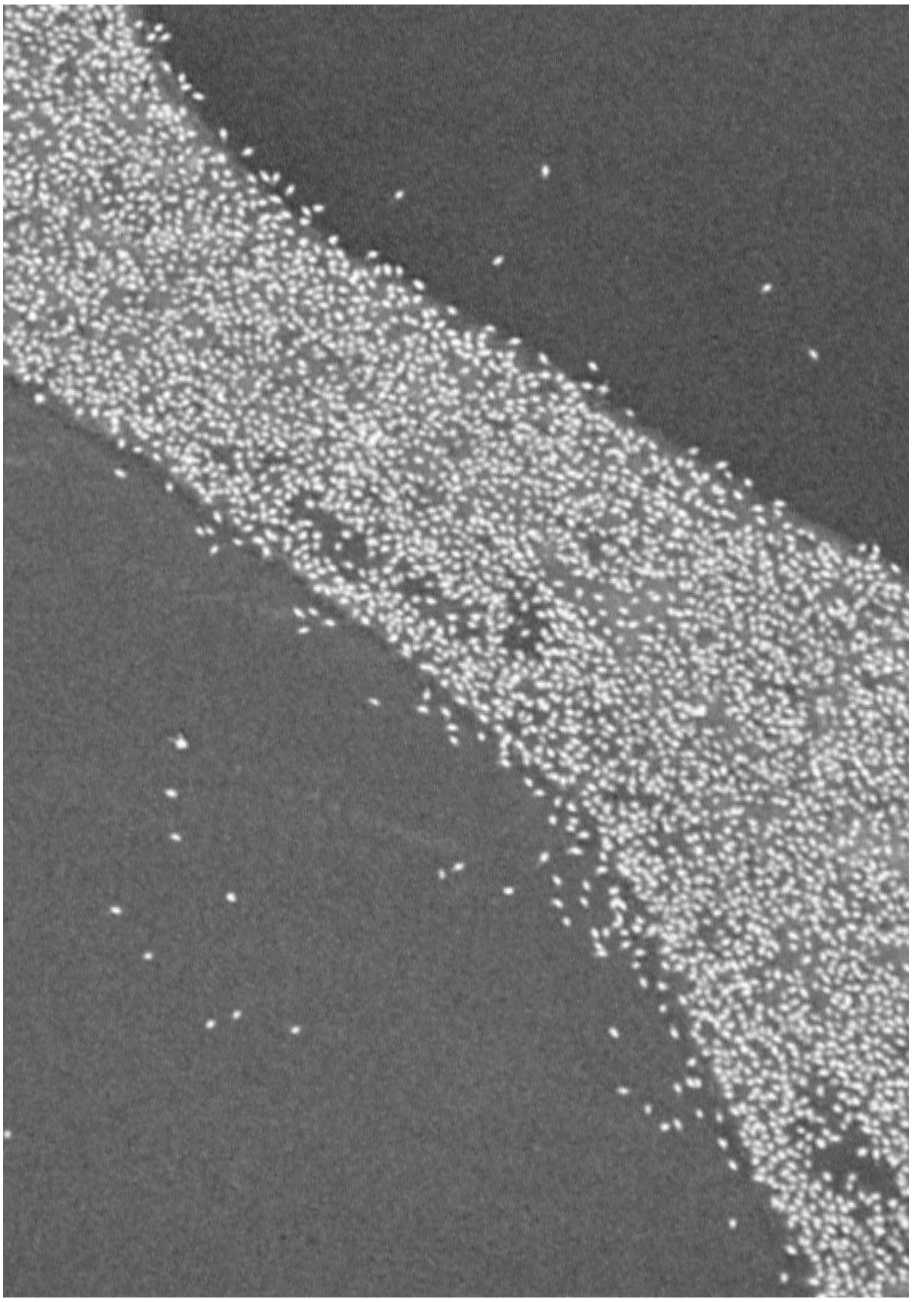

Figure 3: Bird population © Station Biologique Tour du Valat

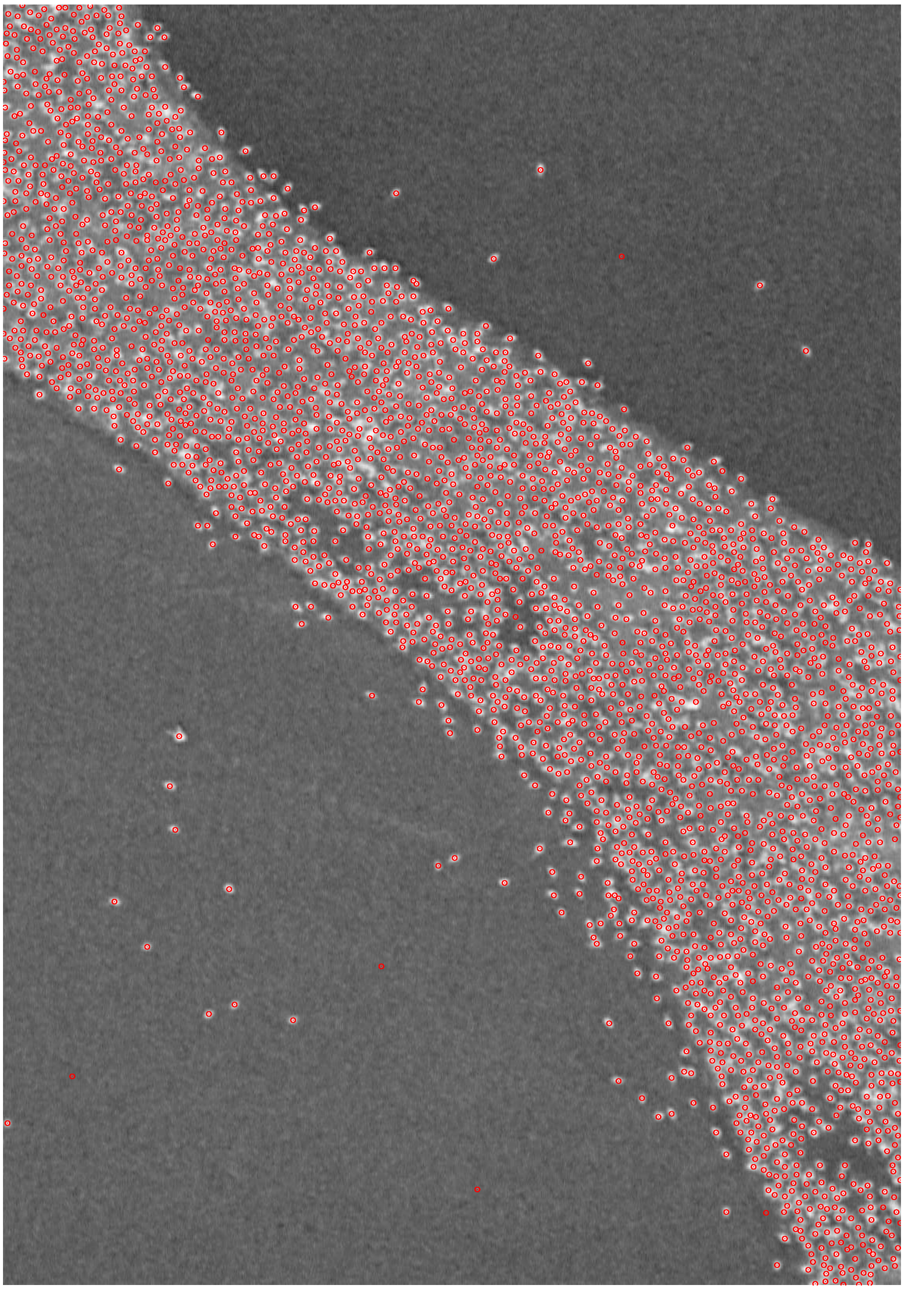

Figure 4: Detected birds from the image shown on figure 3

# Appendix A. Proof of theorem 2.

1) The boundness of $L_\beta$ is obvious, and then we can define the semigroup $T_\beta(t)$ as the exponent of the operator $L_\beta$ using the usual espansion for the exponent. Self-adjointness follows from the detailed balance condition and the boundness of the operator $L_\beta$.
2) Since $L_\beta 1 = 0$ we get $e^{tL_\beta}1 = 1$. The second condition in (14) follows from the analogous property of the operators $P_\beta^n$ and the convergence of the approximation process associated with transition operator $P_\beta$ (see appendix B).
3) Using relations

$$T_\beta(t) \;=\; e^{tL_\beta} \;=\; \lim_{n\to\infty}\left(E+\frac{t}{n}L_\beta\right)^n \tag{34}$$

and

$$\left(E+\frac{t}{n}L_\beta\right)f(\gamma) \;=\; \left(1-\frac{t}{n}\left(\sum_{x\in\gamma}e^{\beta E(x,\gamma\setminus x)}+zV(\gamma)\right)\right)f(\gamma)\;+ \frac{t}{n}\sum_{x\in\gamma}e^{\beta E(x,\gamma\setminus x)}f(\gamma\setminus x)\;+\;\frac{t}{n}\,z\int_{V(\gamma)}f(\gamma\cup y)\;dy \tag{35}$$

we obtain that, for any $t>0$ and for large enough $n$, all coefficients in the decomposition (35) are positive, and moreover,

$$\left|\left(E+\frac{t}{n}L_\beta\right)f(\gamma)\right| \;\le\; \left|\left(E+\frac{t}{n}L_\beta\right)|f|(\gamma)\right| \;\le\; \sup_\gamma|f(\gamma)|.$$

Thus,

$$\sup_\gamma\left|\left(E+\frac{t}{n}L_\beta\right)f(\gamma)\right| \;\le\; \sup_\gamma|f(\gamma)|, \tag{36}$$

and applying inequality (36) $n$ times we still keep the same bound:

$$\sup_\gamma\left|\left(E+\frac{t}{n}L_\beta\right)^n f(\gamma)\right| \;\le\; \sup_\gamma|f(\gamma)| \quad \text{for all large enough} \quad n\in N.$$

Consequently, $T_\beta(t)$ is a contraction semigroup in $B(\Gamma_d(V))$, i.e.

$$\sup_\gamma|(T_\beta(t)\;f)(\gamma)| \;\le\; \sup_\gamma|f(\gamma)|.$$

4) Using (35) we have:

$$\left(E\;+\;\frac{t}{n}L_\beta\right)f \;=\; B_0f\;+\;\frac{t}{n}\;B_+f\;+\;\frac{t}{n}\;B_-f,$$

where

$$(B_0f)(\gamma) \;=\; \left(1-\frac{t}{n}\left(\sum_{x\in\gamma}e^{\beta E(x,\gamma\setminus x)}+zV(\gamma)\right)\right)f(\gamma) \;>\; \left(1-\frac{t}{n}R\right)f(\gamma),$$

$$(B_- f)(\gamma) \;=\; \sum_{x\in\gamma} e^{\beta E(x,\gamma\setminus x)} f(\gamma\setminus x), \quad (B_+ f)(\gamma) \;=\; z \int\limits_{V(\gamma)} f(\gamma\cup x)dx,$$

and

$$R \;=\; \max_{\gamma} \left( \sum_{x\in\gamma} e^{\beta E(x,\gamma\setminus x)} + zV(\gamma) \right) \;<\; \infty.$$

For any given $t>0$, if $n$ is large enough, then $1-\frac{tR}{n}>0$. The following decomposition holds:

$$\left(E \;+\; \frac{t}{n}L_\beta\right)^n f \;=\; \sum_{\substack{(\sigma_1,\ldots,\sigma_n)\\ \sigma_i=0,+,-}} B_{\sigma_1}B_{\sigma_2}\ldots B_{\sigma_n} \left(\frac{t}{n}\right)^{n_+} \left(\frac{t}{n}\right)^{n_-} f, \tag{37}$$

where $n_\pm$ is the number of " $+$ " or correspondingly " $-$ " in the sequence $(\sigma_1,\ldots,\sigma_n)$, and the sum is taken over all sequences $(\sigma_1,\ldots,\sigma_n)$ with $\sigma_i = 0,+,-$. Each term in (37) is non-negative if $f$ is a non-negative function and if $n$ is large enough.

Let us consider two cases. If $f(\emptyset)>0$, then we show that for any $\gamma\neq\emptyset$ the sum

$$\left( \sum_{\substack{(\sigma_1,\ldots,\sigma_n)\\ n_-=k}} B_{\sigma_1}B_{\sigma_2}\ldots B_{\sigma_n} \left(\frac{t}{n}\right)^k f \right)(\gamma) \;\equiv\; (I_n^{(k)}f)(\gamma) \;>\; c_k f(\emptyset), \tag{38}$$

where $|\gamma|=k$ and a constant $c_k>0$ does not depend on $n$. The sum in (38) is taken over all sequences $(\sigma_1,\ldots,\sigma_n)$ free from pluses with $n_-=k$. Indeed, we have for all large enough $n$

$$(I_n^{(k)}f)(\gamma) \;>\; \left(1-\frac{t}{n}R\right)^{n-k} B_-^k(\gamma,\emptyset)\; C_n^k \left(\frac{t}{n}\right)^k f(\emptyset),$$

with the corresponding matrix element of the operator $B_-^k$

$$B_-^k(\gamma,\emptyset) \;=\; k!\; e^{\beta H(\gamma)}.$$

Since for any fixed $t,\ R,\ k$

$$\left(1-\frac{tR}{n}\right)^{n-k} \; C_n^k \; \left(\frac{t}{n}\right)^k \;\to\; \frac{t^k}{k!}\; e^{-tR} \quad \text{as} \quad n\to\infty,$$

we have for large enough $n$

$$\left(1-\frac{tR}{n}\right)^{n-k} \; C_n^k \; \left(\frac{t}{n}\right)^k \;>\; \frac{t^k}{2\;k!}e^{-tR}.$$

Consequently, for any $\gamma$ with $|\gamma|=k$

$$(I_n^{(k)}f)(\gamma) \;>\; \frac{1}{2}\; t^k\; e^{\beta H(\gamma)-tR}\; f(\emptyset) \;\equiv\; c_k\; f(\emptyset), \tag{39}$$

and in the limit (34) as $n \to \infty$ relations (37) - (39) imply that

$$\left(e^{tL_\beta} f\right)(\gamma) \ > \ c_k f(\emptyset) \ > \ 0.$$

Assume now that $f(\emptyset) = 0$ and $f(\gamma) > 0$ on a set $Q \subset \Gamma_d(V, k)$ of a positive measure $\lambda(Q) > 0$. We consider in the sum (37) the following term

$$\sum_{\substack{(\sigma_1,\ldots,\sigma_n) \\ n_+ = k}} B_{\sigma_1} B_{\sigma_2} \ldots B_{\sigma_n} \left(\frac{t}{n}\right)^k \ \equiv \ J_n^{(k)},$$

where the sum is taken over all sequences $(\sigma_1, \ldots, \sigma_n)$ free from minuses with $n_+ = k$. Then as above we have for all large enough $n$

$$(J_n^{(k)} f)(\emptyset) \ > \ \left(1 - \frac{tR}{n}\right)^{n-k} \ C_n^k \ \left(\frac{t}{n}\right)^k \ \int\limits_Q (B_+^k)(\emptyset, \gamma) f(\gamma) d\lambda \ >$$

$$\frac{t^k}{2\ k!} \ e^{-tR} \ \int\limits_Q (B_+^k)(\emptyset, \gamma) f(\gamma) d\lambda.$$

Taking the limit (34) as $n \to \infty$ we have

$$\left(e^{tL_\beta} f\right)(\emptyset) \ > \ \frac{t^k}{2\ k!} \ e^{-tR} \ \int\limits_Q (B_+^k)(\emptyset, \gamma) f(\gamma) d\lambda \ > \ 0.$$

Thus using results of the previous case we get that for any $\gamma$

$$\left(e^{2tL_\beta} f\right)(\gamma) \ > \ 0.$$

Theorem 2 is proved completely.

# Appendix B. Convergence of the approximation processes. Proof of theorem 4.

To prove the convergence of the corresponding semigroup

$$T_{\beta,\delta}(t) = P_{\beta,\delta}^{[\frac{t}{\delta}]} \ \to \ T_\beta(t), \quad \delta \to 0$$

uniformly on bounded intervals of time $t$ we use here the following approximation theorem:

**Theorem [13].** *For $n = 1, 2, \ldots$ let $T_n$ be a linear contraction on a Banach space $\mathfrak{L}$, and let $\delta_n$ be positive numbers. We set:*

$$L_n \ = \ \frac{1}{\delta_n} \ (T_n - E).$$

*Assume that* $\lim_{n\to\infty} \delta_n \ = \ 0.$

*Let* $\{T(t)\}$ *be a strongly continuous contraction semigroup on the Banach space* $\mathfrak{L}$ *with generator* $L$*, and let* $C$ *be a core for* $L$*. Then the following propositions are equivalent:*
*a) for each* $f \in \mathfrak{L}$

$$\|T_n(t)f \ - \ T(t)f\|_{\mathfrak{L}} \ \to \ 0, \quad as \quad \delta_n \to 0$$

*for all* $t \geq 0$ *uniformly on bounded intervals;*
*b) for each* $f \in C$

$$\|L_n f \ - \ Lf\|_{\mathfrak{L}} \ \to \ 0, \quad \text{as} \quad \delta_n \to 0.$$

We denote by $L_{\beta,\delta}$ the generator of the process $T_{\beta,\delta}$ defined by transition probabilities (18) (homogeneous in time):

$$(L_{\beta,\delta} f)(\gamma) \ = \ \frac{1}{\delta_n}\left((P_{\beta,\delta} f)(\gamma) \ - \ f(\gamma)\right) \ = \tag{40}$$

$$\frac{1}{\delta_n}\left(\sum_{\gamma_1\subseteq\gamma} \prod_{x\in\gamma\setminus\gamma_1} (a_x\delta_n) \prod_{x\in\gamma} \frac{1}{1+a_x\delta_n}\right.$$

$$\left.\Xi_\delta^{-1}(\gamma_1) \sum_{k=0}^{\infty} \int\limits_{V_k(\gamma_1)} \frac{(z\delta_n)^k}{k!} \ f(\gamma_1 \cup y_1 \cup \ldots \cup y_k) \ dy_1 \ldots dy_k \ - \ f(\gamma)\right) \ =$$

$$\frac{1}{\delta_n}\left(\Xi_\delta^{-1}(\gamma) \prod_{x\in\gamma} \frac{1}{1+a_x\delta_n} \ f(\gamma) \ - \ f(\gamma)\right) \ +$$

$$\frac{1}{\delta_n} \ \Xi_\delta^{-1}(\gamma\setminus x) \prod_{y\in\gamma} \frac{1}{1+a_y\delta_n} \sum_{x\in\gamma} a_x\delta_n \ f(\gamma\setminus x) \ +$$

$$\frac{1}{\delta_n} \ \Xi_\delta^{-1}(\gamma) \ z\delta_n \prod_{y\in\gamma} \frac{1}{1+a_y\delta_n} \int\limits_{V(\gamma)} f(\gamma\cup\tilde{y}) \ d\tilde{y} \ +$$

$$\frac{1}{\delta_n} \sum_{\tilde{\gamma}\subseteq\gamma} \Xi_\delta^{-1}(\gamma\setminus\tilde{\gamma}) \sum_{k:|\tilde{\gamma}|+k\geq 2} \frac{(z\delta_n)^k}{k!} \prod_{y\in\gamma} \frac{1}{1+a_y\delta_n} \prod_{x\in\tilde{\gamma}} a_x\delta_n$$

$$\int\limits_{V_k(\gamma\setminus\tilde{\gamma})} f((\gamma\setminus\tilde{\gamma}) \cup y_1 \cup \ldots \cup y_k) \ dy_1 \ldots dy_k.$$

We take here as a core $C \ = \ B(\Gamma_d(V))$ a whole set of bounded functions on $\Gamma_V$. Let us consider the following theorem, proved in appendix C:

**Theorem 5.** *Let us denote*

$$\Delta_\delta f \ = \ L_{\beta,\delta} f \ - \ L_\beta f.$$

*Then*

$$\sup_{\gamma} \ |\Delta_\delta f(\gamma)| \ \to \ 0 \quad as \quad \delta \to 0 \tag{41}$$

*for each* $f(\gamma) \in B(\Gamma_V)$.

Finally, relation (41) immediately implies convergence (19) of the semigroups in the uniform norm of the space $\mathcal{L}$ by the above approximation theorem. Theorem 4 is proved.

# Appendix C. Proof of theorem 5.

**Some preliminary expansions.**

**Remark.** We have for any $x \in \gamma$

$$E(x, \gamma \backslash x) \ \le \ b, \tag{42}$$

so that

$$a_x \ \equiv \ e^{\beta E(x,\gamma\backslash x)} \ \le \ e^{\beta b} \tag{43}$$

and let

$$a \ = \ a(\beta) \ = \sup_{\gamma\in\Gamma_d(V)} \ \sup_{x\in\gamma} \ a_x \ < \ \infty.$$

**Lemma 1.** *The normalizing factor* $\Xi_\delta^{-1}(\gamma)$ *from (18) can be written as*

$$\Xi_\delta^{-1}(\gamma) \ = \ 1 \ - \ z\delta|V(\gamma)| \ + \ O\left(z^2\delta^2\right) \quad \text{as} \quad \delta \to 0. \tag{44}$$

**Proof.** Since

$$\Xi_\delta(\gamma) \ = \ 1 \ + \ \sum_{m=1}^{\infty} \frac{(z\delta)^m}{m!} \int\limits_{V_m(\gamma)} dy_1 \dots dy_m \ = \ 1 \ + \ z\delta|V(\gamma)| \ + \ \sum_{m=2}^{\infty} \frac{(z\delta)^m}{m!} \mathcal{V}_m$$

with $\mathcal{V}_m < |V(\gamma)|^m < |V|^m, \ m \ge 2$, we can write

$$\Xi_\delta(\gamma) \ = \ 1 \ + \ z\delta|V(\gamma)| \ + \ O(z^2\delta^2). \tag{45}$$

Thus, (45) implies (44). $\square$

Using the Taylor expansions

$$\ln(1+x) \ = \ x - \frac{x^2}{2}\,\xi', \quad \xi' \in (4/9, 4) \quad \text{as} \quad |x| < \frac{1}{2}, \tag{46}$$

and

$$e^x \ = \ 1 + x + \frac{x^2}{2}\,e^{\xi}, \quad \xi \in (0, x), \tag{47}$$

we have for small enough $\delta$:

$$\frac{1}{1+a_x\delta} = e^{-\ln(1+a_x\delta)} = e^{-a_x\delta+\frac{\xi_1}{2}\ a_x^2\delta^2},$$

$$\prod_{x\in\gamma}\frac{1}{1+a_x\delta} = e^{-\delta\sum\limits_{x\in\gamma} a_x+\frac{\xi_1}{2}\ \delta^2 \sum\limits_{x\in\gamma} a_x^2}. \tag{48}$$

Then using (44), (48) and relation $\frac{\xi_1}{2}\delta^2\sum_{x\in\gamma}a_x^2 = O(a^2\delta^2)$ we have for all small enough $\delta$:

$$\Xi_\delta^{-1}(\gamma)\prod_{x\in\gamma}\frac{1}{1+a_x\delta} - \left(1-\delta\sum_{x\in\gamma}a_x - z\ \delta\ |V(\gamma)|\right) = \tag{49}$$

$$\left(1-z\delta|V(\gamma)|+O\left((z\delta)^2\right)\right)\ e^{-\delta\sum a_x+\delta^2\frac{\xi_1}{2}\sum a_x^2} -$$
$$\left(1\ -\ \delta\sum a_x\ -\ \delta z|V(\gamma)|\right) = O\left(\delta^2(a^2+z^2)\right).$$

We assume here that

$$\delta\ a\ =\ \delta\ a(\beta)\ <\ const, \tag{50}$$

which is of course true for any fixed $\beta$ and small enough $\delta$. Let us write now the expression for $\Delta_\delta f(\gamma)$ using (40) and (13):

$$(\Delta_\delta f)(\gamma)\ =\ \frac{1}{\delta}\left((P_\delta f)(\gamma)\ -\ f(\gamma)\right)\ - \tag{51}$$

$$\sum_{x\in\gamma}a_x(\gamma)\ (f(\gamma\backslash x)-f(\gamma))\ -\ z\int\limits_{V(\gamma)}(f(\gamma\cup y)-f(\gamma))\ dy\ =$$

$$\frac{1}{\delta}\left(\Xi_\delta^{-1}(\gamma)\prod_{x\in\gamma}\frac{1}{1+a_x\delta}\ f(\gamma)\ -\ f(\gamma)\right)\ +$$

$$+\sum_{x\in\gamma}a_x(\gamma)\ f(\gamma)\ +\ z\ |V(\gamma)|\ f(\gamma)\ +$$

$$\frac{1}{\delta}\sum_{x\in\gamma}a_x(\gamma)\delta\ \Xi_\delta^{-1}(\gamma\backslash x)\prod_{y\in\gamma}\frac{1}{1+a_y\delta}\ f(\gamma\backslash x)\ -\ \sum_{x\in\gamma}a_x(\gamma)f(\gamma\backslash x)\ +$$

$$\frac{1}{\delta}\ \Xi_\delta^{-1}(\gamma)\ z\ \delta\prod_{y\in\gamma}\frac{1}{1+a_y\delta}\int\limits_{V(\gamma)}f(\gamma\cup y)\ dy\ -\ z\int\limits_{V(\gamma)}f(\gamma\cup y)\ dy\ +$$

$$\frac{1}{\delta}\sum_{\tilde\gamma\subseteq\gamma}\Xi_\delta^{-1}(\gamma\backslash\tilde\gamma)\sum_{k:|\tilde\gamma|+k\geq 2}\frac{(z\delta)^k}{k!}\prod_{x\in\tilde\gamma}a_x\delta\prod_{y\in\gamma}\frac{1}{1+a_y\delta}$$

$$\int\limits_{V_k(\gamma\backslash\tilde\gamma)}f(\gamma\backslash\tilde\gamma\cup y_1\cup\ldots\cup y_k)\ dy_1\ldots dy_k.$$

**Estimation of the coefficients in (51).**

Let us estimate the terms with $f(\gamma),\ f(\gamma\backslash x),\ f(\gamma\cup y)$ in (51) separately.

Using (49) and boundedness of $f(\gamma)$ we have:

$$\left|\frac{1}{\delta}\left(\Xi_\delta^{-1}(\gamma)\ \prod_{x\in\gamma}\frac{1}{1+a_x\delta}\ f(\gamma)\ -\ f(\gamma)\right)\ +\right.$$

$$\left.\left(\sum_{x\in\gamma}a_x(\gamma)\ +\ z|V(\gamma)|\right)f(\gamma)\right|\ =\ O(\delta(a^2+z^2)),\ \delta\to 0. \tag{52}$$

Analogously, we can estimate the coefficients before $f(\gamma\backslash x)$ and $f(\gamma\cup y)$:

$$\left|\sum_{x\in\gamma}a_x\left(\Xi_\delta^{-1}(\gamma\backslash x)\ \prod_{y\in\gamma}\frac{1}{1+a_y\delta}\ -\ 1\right)\ f(\gamma\backslash x)\right|\ =$$

$$\left|\sum_{x\in\gamma}a_x\left(\left(1-z\delta|V(\gamma\backslash x)|+O(z^2\delta^2)\right)\ e^{-\delta\sum a_x+\frac{\xi_1}{2}\delta^2\sum a_x^2}\ -\ 1\right)\ f(\gamma\backslash x)\right|\ \le$$

$$a|\gamma|\ \left(\delta(a|\gamma|+z|V|)\ +\ \delta^2\ (A_2|\gamma|+A_3z^2|V|^2)\right)\ K_f\ =\ O(\delta a(z+a)). \tag{53}$$

And

$$z\left|\int\limits_{V(\gamma)}\left(\Xi_\delta^{-1}(\gamma)\ \prod_{x\in\gamma}\frac{1}{1+a_x\delta}\ -\ 1\right)f(\gamma\cup y)\ dy\right|\ \le$$

$$z|V|\left(\delta(a|\gamma|+z|V|)\ +\ \delta^2(A_2|\gamma|+A_3z^2|V|^2)\right)\ K_f\ =\ O(\delta z(z+a)). \tag{54}$$

Let us estimate now the last term in (51). Using that

$$\Xi_\delta^{-1}(\gamma)\ \le\ 1,\quad \prod_{y\in\gamma}\frac{1}{1+a_y\delta}\ \le 1,$$

and $\sup_\gamma|f(\gamma)|<K_f$ we have

$$\frac{1}{\delta}\sum_{\tilde\gamma\subseteq\gamma}\Xi_\delta^{-1}(\gamma\backslash\tilde\gamma)\sum_{k:|\tilde\gamma|+k\ge 2}\frac{(z\delta)^k}{k!}\prod_{x\in\tilde\gamma}a_x\delta\prod_{y\in\gamma}\frac{1}{1+a_y\delta}$$

$$\int\limits_{V_k(\gamma\backslash\tilde\gamma)}f(\gamma\backslash\tilde\gamma\cup y_1\cup\ldots\cup y_k)\ dy_1\ldots dy_k\ \le$$

$$\frac{K_f}{\delta} \sum_{\tilde{\gamma} \subseteq \gamma} (a\delta)^{|\tilde{\gamma}|} \sum_{k \geq 0: |\tilde{\gamma}| + k \geq 2} \frac{(z|V|\delta)^k}{k!} \leq$$

$$\frac{K_f}{\delta} a^N \sum_{\substack{m=0,\dots,|\gamma| \\ k \geq 0: m+k \geq 2}} C^m_{|\gamma|} \delta^m \frac{(z|V|\delta)^k}{k!} =$$

$$\frac{K_f}{\delta} a^N \left( e^{z|V|\delta} (1+\delta)^{|\gamma|} - 1 - |\gamma|\delta - z|V|\delta \right) = O(\delta + \delta z + \delta z^2). \tag{55}$$

Here we used that

$$(1+\delta)^{|\gamma|} = e^{|\gamma| \ln(1+\delta)} = e^{\delta|\gamma|} + O(\delta^2) \quad \text{for small } \delta > 0.$$

Finally, from (51), (52), (53), (54), (55) it follows that for any $f(\gamma) \in B(\Gamma_d(V))$

$$\sup_\gamma |\Delta_\delta f(\gamma)| \to 0 \quad \text{as} \quad \delta \to 0,$$

and consequently,

$$\|L_{\beta,\delta} f - L_\beta f\|_{B(\Gamma_d(V))} \to 0 \quad \text{as} \quad \delta \to 0.$$

Theorem 5 is completely proved.

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